SOME INEQUALITIES FOR (α, β)-NORMAL OPERATORS IN HILBERT SPACES

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Abstract. An operator T acting on a Hilbert space is called (α, β)-normal $(0 \leq \alpha \leq 1 \leq \beta)$ if

$$\alpha^2 T^*T \leq TT^* \leq \beta^2 T^*T.$$

In this paper we establish various inequalities between the operator norm and its numerical radius of (α, β)-normal operators in Hilbert spaces. For this purpose, we employ some classical inequalities for vectors in inner product spaces.

1. Introduction

An operator T acting on a Hilbert space $(\mathcal{H}; \langle \cdot, \cdot \rangle)$ is called (α, β)-*normal* $(0 \leq \alpha \leq 1 \leq \beta)$ if

$$\alpha^2 T^*T \leq TT^* \leq \beta^2 T^*T.$$

Then

$$\alpha^2 \langle T^*Tx, x\rangle \leq \langle TT^*x, x\rangle \leq \beta^2 \langle T^*Tx, x\rangle\,,$$

whence

$$\alpha\|Tx\| \leq \|T^*x\| \leq \beta\|Tx\|, \tag{1.1}$$

for all $x \in \mathcal{H}$, namely, both T majorizes T^* and T^* majorizes T. A seminal result of R.G. Douglas [6] (majorization lemma) says that an operator $T \in B(\mathcal{H})$ majorizes an operator $S \in B(\mathcal{H})$ if any one of the following equivalent statements holds:

(i) the range space $\text{ran}(T)$ of T is a subset of $\text{ran}(S)$;

(ii) $TT^* \leq \lambda^2 SS^*$;

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(iii) there exists an operator $R \in B(\mathcal{H})$ such that $T = SR$.

Furthermore, R is the unique operator satisfying

(a) $\|R\| = \inf\{\lambda : TT^* \leq \lambda SS^*\}$;

(b) $\ker(T) = \ker(R)$;

(c) $\mathrm{ran}(R)$ is a subset of the closure $\mathrm{ran}(S^*)^-$ of $\mathrm{ran}(S^*)$.

Analogues of Douglas' majorization lemma for Banach space operators were studied by M.R. Embry [14] (see also [3]). A discussion of the duality between the properties of majorization and range inclusion can be found in [1].

Using the result of Douglas, we observe that T is (α, β)-normal if and only if $\mathrm{ran}(T) = \mathrm{ran}(T^*)$, or, equivalently, $\ker(T) = \ker(T^*)$. It is therefore obvious that invertible, normal and hyponormal operators are (α, β)-normal for some appropriate values of α and β. The matrix $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ in $B(\mathbb{C}^2)$ is an (α, β)-normal with $\alpha = \sqrt{(3-\sqrt{5})/2}$ and $\beta = \sqrt{(3+\sqrt{5})/2}$, which is neither normal nor hyponormal. There are some results which can be applied to our notion in the literature. For instance, one can deduce from [4, Lemma 1] that if z is an eigenvalue of T and z belongs to the topological boundary of the numerical range of T, then $T - z$ is (α, β)-normal for some α and β. There are also some interesting questions in linear algebra concerning (α, β)-normality, see [18].

Another characterization is that T is (α, β)-normal $(0 < \alpha \leq 1 \leq \beta)$ if and only if there are operators $S_1, S_2 \in B(\mathcal{H})$ such that $T = T^*S_1$ and $T = S_2T^*$. Moreover, S_1, S_2 can be chosen in such a way that

$$\|S_1\| = \inf\{\beta \geq 1 : TT^* \leq \beta T^*T\}, \qquad \|S_2\| = \sup\{\alpha > 0 : \alpha T^*T \leq TT^*\}\,.$$

Let T be an (α, β)-normal operator on a (not necessarily finite dimensional) Hilbert space $\mathcal{H}$. Using the fact that $ker(T^*)^\perp = \mathrm{ran}(T)^-$, we observe that $\mathcal{H} = \ker(T) \oplus \mathrm{ran}(T)^-$. Hence T can be represented as a block matrix $\begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix}$, where $C : \mathrm{ran}(T)^- \to \mathrm{ran}(T)^-$ has zero kernel. We can define the pseudo-inverse of T, denoted by T^+, to be the operator on $\mathcal{H}$, which is zero on $\mathrm{ran}(T)^\perp$, and is the inverse to C on $\mathrm{ran}(T)^-$. It is easy to see that T^+

is closed if and only if $\operatorname{ran}(T)$ is closed. The operator pseudo-inverse is a powerful tool in applied mathematics; cf. [2].

Let $(\mathcal{H}; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. The *numerical radius* $w(T)$ of an operator T on $\mathcal{H}$ is given by

$$w(T) = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.2}$$

Obviously, by (1.2), for any $x \in \mathcal{H}$ one has

$$|\langle Tx, x\rangle| \le w(T)\|x\|^2. \tag{1.3}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(\mathcal{H})$ of all bounded linear operators. Moreover, we have

$$w(T) \le \|T\| \le 2w(T) \qquad (T \in B(\mathcal{H})).$$

For other results and historical comments on the numerical radius see [16].

In this paper, we establish various inequalities between the operator norm and its numerical radius of (α, β)-normal operators in Hilbert spaces. For this purpose, we employ some classical inequalities for vectors in inner product spaces due to Buzano, Dunkl–Williams, Dragomir–Sándor, Goldstein–Ryff–Clarke and Dragomir.

2. Inequalities Involving Numerical Radius

In this section we study some inequalities concerning the numerical radius and norm of (α, β)-normal operators. Our first result reads as follows, see also [11]:

Theorem 2.1. *Let $T \in B(\mathcal{H})$ be an (α, β)-normal operator. Then*

$$(\alpha^{2r} + \beta^{2r})\|T\|^2 \le \begin{cases} 2\beta^r w(T^2) + r^2\beta^{2r-2}\|\beta T - T^*\|^2, & \text{if } r \ge 1, \\ \\ 2\beta^r w(T^2) + \|\beta T - T^*\|^2, & \text{if } r < 1. \end{cases} \tag{2.1}$$

Proof. We use the following inequality for vectors in inner product spaces due to Goldstein, Ryff and Clarke [15]:

$$\|a\|^{2r} + \|b\|^{2r} - 2\|a\|^r\|b\|^r \cdot \frac{\mathrm{Re}\langle a, b\rangle}{\|a\|\,\|b\|} \le \begin{cases} r^2\|a\|^{2r-2}\|a-b\|^2 & \text{if } r \ge 1, \\ \\ \|b\|^{2r-2}\|a-b\|^2 & \text{if } r < 1, \end{cases} \tag{2.2}$$

provided $r \in \mathbb{R}$ and $a, b \in H$ with $\|a\| \ge \|b\|$.

Suppose that $r \ge 1$. Let $x \in H$ with $\|x\| = 1$. Noting to (1.1) and applying (2.2) for the choices $a = \beta Tx$, $b = T^*x$ we get

$$\begin{aligned} &\|\beta Tx\|^{2r} + \|T^*x\|^{2r} - 2\|\beta Tx\|^{r-1}\|\,\|T^*x\|^{r-1}\,\mathrm{Re}\langle \beta Tx, T^*x\rangle \\ &\qquad\qquad \le r^2\|\beta Tx\|^{2r-2}\|\beta Tx - T^*x\|^2 \end{aligned} \tag{2.3}$$

for any $x \in H$, $\|x\| = 1$ and $r \ge 1$. Using (1.1) and (2.3) we get

$$\begin{aligned} &(\alpha^{2r} + \beta^{2r})\|Tx\|^{2r} \\ &\le 2\beta^r\|Tx\|^{r-1}\|T^*x\|^{r-1}|\langle T^2x, x\rangle| + r^2\beta^{2r-2}\|Tx\|^{2r-2}\|\beta Tx - T^*x\|^2. \end{aligned} \tag{2.4}$$

Taking the supremum in (2.4) over $x \in H$, $\|x\| = 1$, we deduce

$$(\alpha^{2r} + \beta^{2r})\|T\|^{2r} \le 2\beta^r\|T\|^{2r-2}\|T^*\|^{r-1}w(T^2) + r^2\beta^{2r-2}\|T\|^{2r-2}\|\beta T - T^*\|^2,$$

which is the first inequality in (2.1). If $r < 1$, then one can similarly prove the second inequality in (2.1). □

Theorem 2.2. *Let $T \in B(\mathcal{H})$ be an (α, β)-normal operator. Then*

$$w(T)^2 \le \frac{1}{2}\left[\beta\|T\|^2 + w(T^2)\right]. \tag{2.5}$$

Proof. The following inequality is known in the literature as the *Buzano inequality* [5]:

$$|\langle a, e\rangle\langle e, b\rangle| \le \frac{1}{2}(\|a\|\,\|b\| + |\langle a, b\rangle|), \tag{2.6}$$

for any a, b, e in $\mathcal{H}$ with $\|e\| = 1$.

Let $x \in H$ with $\|x\| = 1$. Put $e = x, a = Tx, b = T^*x$ in (2.6) to get

$$|\langle Tx, x\rangle\langle x, T^*x\rangle| \le \frac{1}{2}(\|Tx\|\,\|T^*x\| + |\langle Tx, T^*x\rangle|)$$
$$\le \frac{1}{2}(\beta\|Tx\|^2 + |\langle T^2x, x\rangle|).$$

Taking the supremum over $x \in H$, $\|x\| = 1$, we obtain (2.5). □

Theorem 2.3. *Let $T \in B(\mathcal{H})$ be an (α,β)-normal operator and $\lambda \in \mathbb{C}$. Then*

$$\alpha\|T\|^2 \le w(T^2) + \frac{2\beta\|T - \lambda T^*\|^2}{(1 + |\lambda|\alpha)^2}. \tag{2.7}$$

Proof. Using the *Dunkl–Williams inequality* [13]

$$\frac{1}{2}(\|a\| + \|b\|)\left\|\frac{a}{\|a\|} - \frac{b}{\|b\|}\right\| \le \|a - b\| \qquad (a, b \in H \setminus \{0\})$$

we get

$$2 - 2\cdot\frac{\operatorname{Re}\langle a, b\rangle}{\|a\|\|b\|} = \left\|\frac{a}{\|a\|} - \frac{b}{\|b\|}\right\|^2 \le \frac{4\|a - b\|^2}{(\|a\| + \|b\|)^2} \qquad (a, b \in H \setminus \{0\})$$

whence

$$\|a\|\|b\| \le \frac{2\|a\|\,\|b\|\,\|a - b\|^2}{(\|a\| + \|b\|)^2} + |\langle a, b\rangle| \qquad (a, b \in H \setminus \{0\}).$$

Put $a = Tx$ and $b = \lambda T^*$ to get

$$\|Tx\|\,\|T^*x\| \le |\langle T^2x, x\rangle| + \frac{2\|Tx\|\,\|T^*x\|\,\|Tx - \lambda T^*x\|^2}{(\|Tx\| + |\lambda|\,\|T^*x\|)^2}$$

so that

$$\alpha\|Tx\|^2 \le |\langle T^2x, x\rangle| + \frac{2\beta\|Tx\|^2\|Tx - \lambda T^*x\|^2}{(\|Tx\| + |\lambda|\alpha\,\|Tx\|)^2}$$
$$\le |\langle T^2x, x\rangle| + \frac{2\beta\|(T - \lambda T^*)x\|^2}{(1 + |\lambda|\alpha)^2}. \tag{2.8}$$

Taking the supremum in (2.8) over $x \in H$, $\|x\| = 1$, we get the desired result (2.7). □

Theorem 2.4. *Let $T \in B(\mathcal{H})$ be an (α,β)-normal operator and $\lambda \in \mathbb{C}\backslash\{0\}$. Then*

$$\left[\alpha^2 - \left(\frac{1}{|\lambda|} + \beta\right)^2\right] \|T\|^4 \le w(T^2). \tag{2.9}$$

Proof. We apply the following reverse of the quadratic Schwarz inequality obtained by Dragomir in [10]

$$(0 \le)\|a\|^2\|b\|^2 - |\langle a,b\rangle|^2 \le \frac{1}{|\lambda|^2}\|a\|^2\|a-\lambda b\|^2 \tag{2.10}$$

provided $a, b \in H$ and $\lambda \in \mathbb{C}\backslash\{0\}$.

Set $a = Tx, b = T^*x$ in (2.10), to get

$$\begin{aligned}\alpha^2\|Tx\|^4 &\le |\langle Tx, T^*x\rangle|^2 + \frac{1}{|\lambda|^2}\|Tx\|^2\|Tx-\lambda T^*x\|^2\\ &\le |\langle T^2x, x\rangle|^2 + \frac{1}{|\lambda|^2}\|Tx\|^2(1+|\lambda|\beta)^2\|Tx\|^2\end{aligned}$$

whence

$$\left[\alpha^2 - \left(\frac{1}{|\lambda|} + \beta\right)^2\right] \|Tx\|^4 \le |\langle T^2x, x\rangle|^2. \tag{2.11}$$

Taking the supremum in (2.11) over $x \in H$, $\|x\| = 1$, we get the desired result (2.9). □

Theorem 2.5. *Let $T \in B(\mathcal{H})$ be an (α,β)-normal operator, $r \ge 0$ and $\lambda \in \mathbb{C}\setminus\{0\}$. If $\|\lambda T^* - T\| \le r$ and $\frac{r}{|\lambda|} \le \inf\{\|T^*x\| : \|x\| = 1\}$, then*

$$\alpha^2\|T\|^4 \le w(T^2)^2 + \frac{r^2}{|\lambda|^2}\|T\|^2. \tag{2.12}$$

Proof. We use the following reverse of the Schwarz inequality obtained by Dragomir in [8] (see also [9, p. 20]):

$$(0 \le)\,\|y\|^2\,\|a\|^2 - [\mathrm{Re}\langle y,a\rangle]^2 \le r^2\|y\|^2, \tag{2.13}$$

provided $\|y-a\| \le r \le \|a\|$.

By the assumption of theorem $\|Tx - \lambda T^*x\| \le r \le \|\lambda T^*x\|$. Setting $a = \lambda T^*x$ and $y = Tx$, with $\|x\| = 1$ in (2.13) we get

$$\|Tx\|^2\,\|\lambda T^*x\|^2 \le [\mathrm{Re}\langle Tx, \lambda T^*x\rangle]^2 + r^2\|Tx\|^2$$

whence

$$\alpha^2|\lambda|^2\|Tx\|^4 \le |\lambda|^2|\langle T^2x, x\rangle|^2 + r^2\|Tx\|^2. \tag{2.14}$$

Taking the supremum in (2.14) over $x \in H$, $\|x\| = 1$, we get the desired result (2.12). □

Finally, the following result that is less restrictive for the involved parameters r and λ (from the above theorem) may be stated as well:

Theorem 2.6. *Let $T \in B(\mathcal{H})$ be an (α,β)-normal operator, $r \ge 0$ and $\lambda \in \mathbb{C}\setminus\{0\}$. If $\|\lambda T^* - T\| \le r$, then*

$$\alpha\|T\|^2 \le w(T^2) + \frac{r^2}{2|\lambda|}. \tag{2.15}$$

Proof. We use the following reverse of the Schwarz inequality obtained by Dragomir in [7] (see also [9, p. 27]):

$$(0 \le)\ \|y\|\,\|a\| - \operatorname{Re}\langle y, a\rangle \le \frac{1}{2}r^2, \tag{2.16}$$

provided $\|y - a\| \le r$.

Setting $a = \lambda T^*x$ and $y = Tx$, with $\|x\| = 1$ in (2.16) we get

$$\|Tx\|\,\|\lambda T^*x\| \le |\langle Tx, \lambda T^*x\rangle| + \frac{1}{2}r^2$$

which gives

$$\alpha\|Tx\|^2 \le |\langle T^2x, x\rangle| + \frac{1}{2|\lambda|}r^2.$$

Now, taking the supremum over $\|x\| = 1$ in this inequality, we get the desired result (2.15) □

3. Inequalities Involving Norms

Our first result in this section reads as follows.

Theorem 3.1. *Let $T \in B(\mathcal{H})$ be an (α,β)-normal operator. If $p \ge 2$, then*

$$2(1+\alpha^p)\|T\|^p \le \frac{1}{2}(\|T+T^*\|^p + \|T-T^*\|^p). \tag{3.1}$$

In general, for each $T \in B(\mathcal{H})$ and $p \ge 2$ we have

$$\left\|\frac{T^*T+TT^*}{2}\right\|^{p/2} \le \frac{1}{4}(\|T+T^*\|^p + \|T-T^*\|^p). \tag{3.2}$$

Proof. We use the following inequality obtained by Dragomir and Sándor in [12] (see also [17, p. 544]):

$$\|a+b\|^p + \|a-b\|^p \geq 2(\|a\|^p + \|b\|^p) \tag{3.3}$$

for any $a, b \in H$ and $p \geq 2$.

Now, if we choose $a = Tx$, $b = T^*x$ in (3.3), then we get

$$\|Tx + T^*x\|^p + \|Tx - T^*x\|^p \geq 2(\|Tx\|^p + \|T^*x\|^p), \tag{3.4}$$

whence

$$\|Tx + T^*x\|^p + \|Tx - T^*x\|^p \geq 2(\|Tx\|^p + \alpha^p\|Tx\|^p), \tag{3.5}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum in (3.5) over $x \in H$, $\|x\| = 1$, we get the desired result (3.1).

Now for the general case $T \in B(\mathcal{H})$, observe that

$$\|Tx\|^p + \|T^*x\|^p = (\|Tx\|^2)^{\frac{p}{2}} + (\|T^*x\|^2)^{\frac{p}{2}} \tag{3.6}$$

and by applying the elementary inequality:

$$\frac{a^q + b^q}{2} \geq \left(\frac{a+b}{2}\right)^q, \quad a, b \geq 0 \text{ and } q \geq 1$$

we have

$$\begin{aligned}(\|Tx\|^2)^{\frac{p}{2}} + (\|T^*x\|^2)^{\frac{p}{2}} &\geq 2^{1-\frac{p}{2}}(\|Tx\|^2 + \|T^*x\|^2)^{\frac{p}{2}} \\ &= 2^{1-\frac{p}{2}}[\langle Tx, Tx\rangle + \langle T^*x, T^*x\rangle]^{\frac{p}{2}} \\ &= 2^{1-\frac{p}{2}}[\langle (T^*T + TT^*)x, x\rangle]^{\frac{p}{2}}. \end{aligned} \tag{3.7}$$

Combining (3.4) with (3.7) and (3.6) we get

$$\frac{1}{4}[\|Tx - T^*x\|^p + \|Tx + T^*x\|^p] \geq \left|\left\langle \left(\frac{T^*T + TT^*}{2}\right) x, x \right\rangle\right|^{p/2} \tag{3.8}$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $x \in H$, $\|x\| = 1$, and taking into account that

$$w\left(\frac{T^*T + TT^*}{2}\right) = \left\|\frac{T^*T + TT^*}{2}\right\|,$$

we deduce the desired result (3.2). $\square$

Theorem 3.2. *Let $T \in B(\mathcal{H})$ be an (α,β)-normal operator. If $p \in (1,2)$ and $\lambda,\mu \in \mathbb{C}$, then*

$$[(|\lambda| + \beta|\mu|)^p + \max\{|\lambda| - |\mu|\beta, \alpha|\mu| - |\lambda|\}] \, \|T\|^p \\ \leq \|\lambda T + \mu T^*\|^p + \|\lambda T - \mu T^*\|^p. \quad (3.9)$$

Proof. We use the following inequality obtained by Dragomir and Sándor in [12] (see also [17, p. 544])

$$(\|a\| + \|b\|)^p + |\, \|a\| - \|b\| \,|^p \leq ||a+b\|^p + \|a-b\|^p, \qquad (3.10)$$

for any $a, b \in H$ and $p \in (1,2)$.

Put $a = \lambda T x$, $b = \mu T^* x$ in (3.10) to obtain

$$(\|\lambda Tx\| + \|\mu T^*x\|)^p + |\|\lambda Tx\| - \|\mu T^*x\||^p \\ \leq \|\lambda Tx + \mu T^*x\|^p + \|\lambda Tx - \mu T^*x\|^p,$$

whence

$$(|\lambda| + |\mu|\alpha)^p \|Tx\|^p + (\max\{|\lambda| - |\mu|\beta, \alpha|\mu| - |\lambda|\}) \, \|Tx\|^p \\ \leq \|\lambda Tx + \mu T^*x\|^p + \|\lambda Tx - \mu T^*x\|^p, \quad (3.11)$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum in (3.11) over $x \in H$, $\|x\| = 1$, we get the desired result (3.9). □

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